UDC 532.516

SMALL PERTURBATIONS IN A STREAM ISSUING FROM A SLIT

PMM Vol. 36, №4, 1972, pp. 641-646
N. S. KOZIN
(Moscow)
(Received September 25, 1971)

Stability of a stream issuing from a slit-like opening between plane walls under the action of small potential perturbations is considered.

1. Steady flow. We consider a plane problem for a stream of an ideal fluid issuing from a slit-like opening in a vessel whose walls are half-planes forming an angle of 2α, $0 \leqslant \alpha \leqslant \pi$. A solution of the corresponding stationary problem for an incompressible fluid is given in e.g. [1]. It is defined by mapping the hodograph plane on the flow potential plane

$$\xi = \ln[\zeta^\beta(1-\zeta^{2\beta})^{-1}], \quad \xi = \varphi_0 + i\psi_0, \quad \zeta = u_0 - iv_0, \quad \beta = \pi/2\alpha \tag{1.1}$$

Here ξ is the complex potential and ζ is the complex velocity. The flow hodograph is represented by the sector $|\zeta| \leqslant 1$, $|\arg\zeta| \leqslant \alpha$ and the potential region by the strip $|\psi_0| \leqslant \pi/2$. The rays $\varphi_0 < -\ln 2$ and $|\psi_0| = \pi/2$ correspond to the rigid walls in the physical plane $z_0 = x_0 + iy_0$ and the rays $\varphi_0 > -\ln 2$ and $|\psi_0| = \pi/2$ correspond to the free surface.

2. Perturbation equations. Let the flow (1.1) be subjected at the instant $t = 0$ to small perturbations such that the velocity field remains potential, and let the potential of the perturbed flow be represented by

$$f(z_0, 0) = \xi + \varepsilon f_1(z_0, 0) + O(\varepsilon^2)$$

Here ε is a small real parameter, while the function $f_1(z_0, 0)$ is analytic and bounded in the region D_0 of the steady flow. At $t = 0$ the perturbed flow is potential, therefore at $t > 0$ the perturbations satisfy the following system of equations:

$$\operatorname{Re}\frac{\partial f}{\partial t} + \frac{w\bar{w}}{2} + \frac{p}{\rho} = 0, \qquad w = \frac{df}{dz} \tag{2.1}$$

Here $f(z, t)$ denotes the complex potential, $w(z, t)$ is the complex velocity, p is pressure, ρ is the constant density of the fluid and $z = x + iy$ is the complex variable in the region D_t on the physical plane of the flow.

Since the initial perturbation is small, it is expedient to assume that D_t differs little from D_0. Following [2] we linearize (2.1) using the conformal mapping $D_0 \to D_t$

$$z = z(z_0, t), \qquad z(\pm\infty, t) = \pm\infty \tag{2.2}$$

The mapping (2.2) exists, since the regions D_0 and D_t are not too dissimilar, and we have $z = z_0 + \varepsilon z_1(z_0, t)$, $z_1 = z_0 O(1)$.

Let us set

$$p(z,t) = p_0(z_0) + \varepsilon p_1(z_0, t), \quad w(z,t) = \zeta(z_0) + \varepsilon w_1(z_0, t)$$
$$f(z,t) = \xi(z_0) + \varepsilon[f_1(z_0, t) + \zeta z_1(z_0, t)]$$

Then taking the first order terms in ε, we obtain from (2.2)

$$p_1 = -\rho\,\mathrm{Re}\left(\frac{\partial f_1}{\partial t} + \bar{\zeta} w_1\right), \qquad w_1 = \zeta\left(\frac{\partial f_1}{\partial \xi} + \zeta s(\xi)\, z_1\right) \tag{2.3}$$
$$s(\xi) = \beta^{-1}(1 + 4e^{2\xi})^{-1/2}$$

The following conditions hold at the free surface:

$$p = \mathrm{const}, \qquad \left(\frac{dz}{dt}\right)_n = (\bar{w})_n \tag{2.4}$$

Linearizing (2.4) we find

$$\mathrm{Re}\left(\frac{\partial f_1}{\partial t} + \frac{w_1}{\zeta}\right) = 0, \quad \mathrm{Im}\left(\zeta\frac{\partial z_1}{\partial t} + \zeta\frac{\partial z_1}{\partial \xi} + \frac{w_1}{\zeta}\right) = 0 \tag{2.5}$$

where ξ belongs to the rays $\varphi_0 > -\ln 2$ and $|\psi_0| = \pi/2$. At the rigid walls we have

$$\mathrm{Im}\,(w_1/\zeta) = 0, \qquad \mathrm{Im}\,(\zeta z_1) = 0 \tag{2.6}$$

Relations (2.6) can be used to show that (2.5) hold at the rigid walls. By virtue of the analyticity of w_1 and z_1 in D_0 and by (2.3), the function w_1/ζ is bounded when $\xi \to \infty$ and has integrable singularity at the points $\xi = -\ln 2 \pm i\pi/2$. Assuming that $\zeta\,\partial z_1/\partial t$ and $\partial f_1/\partial t$ are bounded in D_0, the assumption corresponding to a smooth variation in time of the hydrodynamic functions and of D_t, we find that the fact that (2.5) holds at the boundary of the strip $|\psi_0| \leqslant \pi/2$ implies that equations

$$\frac{\partial f_1}{\partial t} + \frac{w_1}{\zeta} = ia(t), \qquad \zeta\frac{\partial z_1}{\partial t} + \zeta\frac{\partial z_1}{\partial \xi} + \frac{w_1}{\zeta} = b(t) \tag{2.7}$$

hold everywhere in $|\psi_0| \leqslant \pi/2$. The arbitrary functions $a(t)$ and $b(t)$ appearing in the right-hand sides of (2.7) can be assumed equal to zero. Indeed, the potential is determined with the accuracy of up to the function of time and the mapping $D_0 \to D_t$ must be normalized. Using (2.3) to eliminate the potential f_1 from (2.7) with $a(t) = b(t) = 0$ and setting $w_2 = w_1/\zeta^2$, we obtain

$$\frac{\partial}{\partial t}\begin{Vmatrix} w_2 \\ z_1 \end{Vmatrix} + \begin{Vmatrix} 1 & s(\xi) \\ 0 & 1 \end{Vmatrix}\frac{\partial}{\partial \xi}\begin{Vmatrix} w_2 \\ z_1 \end{Vmatrix} + \begin{Vmatrix} 2s(\xi) & 0 \\ 1\,(\xi) & 0 \end{Vmatrix}\begin{Vmatrix} w_2 \\ z_1 \end{Vmatrix} = 0 \tag{2.8}$$

Using (2.7) we can also obtain a formula for computing p_1 in terms of the perturbed velocity: $p_1 = \rho\,\mathrm{Re}\,[\zeta w_2(1 - |\zeta|^2)]$.

3. Initial values. The initial values are specified for the system (2.8) for $t = 0$ in the form $w_2(\xi, 0) = Q(\xi)$ and $z_1(\xi, 0) = P(\xi)$. The functions $\rho(\xi)$ and $Q(\xi)$ are assumed analytic in the strip $|\psi_0| \leqslant \pi/2$. Moreover, they must satisfy the conditions at the walls ("the compatibility conditions"), i.e.

$$\mathrm{Im}\,(\zeta P(\xi)) = \mathrm{Im}\,(\zeta Q(\xi)) = 0, \quad |\psi_0| = \pi/2, \quad \varphi_0 < -\ln 2 \tag{3.1}$$

We can describe the functions $P(\xi)$ and $Q(\xi)$ effectively by considering them in the hodograph plane. Then the condition (3.1) will hold on the segments $\zeta = re^{\pm i\alpha}$, $0 \leqslant r \leqslant 1$. Applying in the plane $\zeta_1 = \zeta^3$ the Schwartz symmetry principle and the theorem on removable singularity, we obtain

$$Q = \sum_{k \geqslant n} i^k Q_k\,[r(\xi)]^{k-\frac{1}{3}}, \qquad P = \sum_{k \geqslant n} i^k P_k\,[r(\xi)]^{k-\frac{1}{3}} \tag{3.2}$$
$$r(\xi) = \sqrt{1 + e^{-2\xi}/4} - e^{-\xi}/2, \qquad n = [2\alpha/\pi] + 1$$

Here Q_k and P_k are real numbers. The choice of the number n is governed by the condition that $p_1 \to 0$ as $\zeta \to 0$ $(\xi \to -\infty)$, i.e. by the condition that the pressure at infinity doesn't vary.

Thus the problem of propagation of perturbations has been reduced to solving the Cauchy problem for the system (2.8) with initial values (3.2) and the problem of stability, to that of determining the rate of growth of solutions of (2.8) with time.

4. The limiting case of $\alpha = 0$. The case when the fluid flows out of a pipe with parallel walls is particularly simple, and the solution of (2.8) and (3.2) can in this case be obtained in the explicit form. The stationary flow is given by the formulas $\xi = z_0$, $\zeta = 1$, $p_0 = \text{const}$. The perturbation equations are obtained from (2.8) by performing the limiting passage with $\alpha \to 0$

$$\frac{\partial w_2}{\partial t} + \frac{\partial w_2}{\partial z_0} = 0, \qquad \frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial z_0} + w_2 = 0 \tag{4.1}$$

and the initial conditions

$$w_2(z_0, 0) = \sum_{k\geqslant 1} i^k Q_k [r(\xi)]^k, \qquad z_1(z_0, 0) = \sum_{k\geqslant 1} i^k P_k [r(\xi)]^k \tag{4.2}$$

follow from (3.2). The solution of (4.1) with initial values (4.2) has the form

$$w_2(z_0, t) = \sum_{k\geqslant 1} i^k Q_k [r(z_0 - t)]^k$$

$$z_1(z_0, t) = \sum_{k\geqslant 1} i^k (P_k - tQ_k) [r(z_0 - t)]^k \tag{4.3}$$

From (4.3) it follows that the flow is unstable and the perturbations of the free surface increase linearly with t. The solution obtained has a physical sense only for a finite period of time. Indeed, when $w_2(z_0, 0) \neq 0$, the upper and lower boundaries of the stream will have to intersect at a certain instant of time, and this corresponds to the break-up of the stream into separate drops. The formulas (4.3) also show that the Fourier method cannot be applied to the problem (4.1), (4.2). Indeed, (4.1), (4.2) have no solution of the form $e^{\lambda t} F_\lambda(z_0)$. This fact indicates the absence of natural oscillations in the flow and the "drift" character of the propagation of perturbations.

Before constructing a solution to the problem (2.8), (3.2), we should make the following remarks:

1. The system (2.8) describing the perturbations has multiple characteristics and is of the "Jordan cell" type. As we know, the Cauchy problem for such a system is not correct in the Hadamard sense. The correctness of the statement of the problem is ensured in this case by the rigorous requirement that the initial values are analytic in the region of the flow.

2. The characteristics of the system (2.8) are represented by the streamlines of the corresponding steady flow. From this point of view the streamlines are trajectories along which small potential perturbations are propagated (we know that the vortical perturbations are also translated along the streamlines).

3. Use of the complex variable makes it possible to reduce the solution of the problem, after separating the time-dependence, to integration of a system of ordinary differential equations (instead of the partial derivatives usually encountered in the plane problems).

4. Since the asymptotics of the solutions of (2. 8) and (4. 1) must become identical when $\xi \to \infty$, we should expect the solutions of (2. 8) to increase at least linearly with t. The solution of (2. 8) should be sought in the form of an expression, not in terms of the elementary wave solutions, but in terms of the elementary solutions with the initial values given by

$$
\begin{aligned}
w_{2,k}(\xi, 0) &= [r(\xi)]^{k-\frac{1}{\beta}} i^k; \ 0 \\
z_1(\xi, 0) &= 0; \qquad [r(\xi)]^{k-\frac{1}{\beta}} i^k
\end{aligned}
\tag{4.4}
$$

Generally speaking, use of the Fourier method to solve the problem of stability without analyzing the class of initial perturbations may lead to omissions, and even errors. Examples of such omissions are given in [3]. Use of the Fourier method by the autnors of [2, 4] led to an erroneous result. The spectrum of purely real eigenvalues points to the absence of wave solutions.

5. Constructing the solution. Let us consider the system of ordinary differential equations obtained by applying the Laplace transform to (2. 8) with the initial values (3. 2)

$$
\frac{d}{d\xi}\left\|\begin{matrix} w_\lambda \\ z_\lambda \end{matrix}\right\| + \left\|\begin{matrix} \lambda + s & -\lambda s \\ 1 & \lambda \end{matrix}\right\| \left\|\begin{matrix} w_\lambda \\ z_\lambda \end{matrix}\right\| = \left\|\begin{matrix} Q - sP \\ P \end{matrix}\right\|
\tag{5.1}
$$

$$
w_\lambda = \int_0^\infty w_2 e^{-\lambda t} dt, \qquad z_\lambda = \int_0^\infty z_1 e^{-\lambda t} dt
$$

$$
\lambda = \sigma + i\tau, \qquad \sigma \geqslant \sigma_0 > 0
$$

To show that a solution of (2. 8) with the initial values (3. 2) exists, it is sufficient to show that a particular solution of (5. 1) exists, for which the estimate

$$
|w_\lambda| = O(|\lambda|^{-1}), \qquad |z_\lambda| = O(|\lambda|^{-1})
\tag{5.2}
$$

holds in the strip $|\psi_0| \leqslant \pi/2$ for sufficiently large $|\lambda|$. Then the formulas expressing the solution of the problem (2. 8), (3. 2) have the form

$$
\left\|\begin{matrix} w_2 \\ z_1 \end{matrix}\right\| = v.\ p.\ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left\|\begin{matrix} w_\lambda \\ z_\lambda \end{matrix}\right\| e^{\lambda t} d\lambda
\tag{5.3}
$$

To simplity the process of constructing the solution, let us consider the case of the initial values $P(\xi) = 0$ and $Q(\xi) \neq 0$. The case of $P(\xi) \neq 0$ and $Q(\xi) = 0$ is treated in an identical manner. It should be noted that if the estimate $|w_\lambda| <$ const $|\lambda|^{-1}$ holds when $\sigma \geqslant \sigma_0$, then

$$
z_\lambda = -\int_{-\infty}^{\xi} w_\lambda e^{\lambda(\eta - \xi)} d\eta
$$

and the estimate $|z_\lambda| <$ const $|\lambda|^{-1}$ follows. Setting $P = 0$ and passing in (5. 1) to the variable $\zeta_1 = \zeta^\beta$ and to the unknown functions $w_{\lambda,\alpha} = \zeta_1^{1/\beta} w_\lambda$ and $z_{\lambda,\alpha} = \zeta_1^{1/\beta} z_\lambda$ we obtain

$$
\zeta_1 \frac{d}{d\zeta_1}\left\|\begin{matrix} w_{\lambda,\alpha} \\ z_{\lambda,\alpha} \end{matrix}\right\| = \left\|\begin{matrix} -\lambda c(\zeta_1), & \dfrac{\lambda}{\beta} \\ -c(\zeta_1), & \dfrac{1}{\beta} - \lambda c(\zeta_1) \end{matrix}\right\| \left\|\begin{matrix} w_{\lambda,\alpha} \\ z_{\lambda,\alpha} \end{matrix}\right\| + \left\|\begin{matrix} R(\zeta_1) \\ 0 \end{matrix}\right\|
\tag{5.4}
$$

$$
c(\zeta_1) = \frac{1 + \zeta_1^2}{1 - \zeta_1^2}, \qquad R(\zeta_1) = c(\zeta_1) \sum_{k=n}^{\infty} i^k Q_k \zeta_1^k
$$

The system (5.4) has an essential singularity at $\zeta_1 = 0$. Following [5] (p.198) we can show that (5.4) has a unique solution analytic within the circle $|\zeta_1| < 1$. Its Taylor expansion has the form

$$\left\| \begin{matrix} w_{\lambda,\alpha} \\ z_{\lambda,\alpha} \end{matrix} \right\| = \sum_{k=n}^{\infty} \left\| \begin{matrix} a_k(\lambda) \\ b_k(\lambda) \end{matrix} \right\| \zeta_1^k \tag{5.5}$$

The coefficients $a_k(\lambda)$ and $b_k(\lambda)$ are found from the following recurrent relations:

$$a_n = \frac{r_n(n + \lambda c_0 - 1/3)}{\Delta_n(\lambda)}, \qquad b_n = -\frac{r_n c_0}{\Delta_n(\lambda)}$$

$$a_k = \Delta_k^{-1}\left[r_k\left(k + \lambda c_0 - \frac{1}{3}\right) - \lambda(k + \lambda c_0)\sum_{j=1}^{k-n} c_j\left(a_{k-j} - \frac{\lambda^2}{\beta} b_{k-j}\right)\right], \quad k > n$$

$$b_k = -\Delta_k^{-1}\left[c_0 r_k + k\sum_{j=1}^{k-n} c_j a_{k-j} + \lambda(k + \lambda c_0)\sum_{j=1}^{k-n} c_j b_{k-j}\right], \quad k > n$$

$$\Delta_k(\lambda) = (k + \lambda c_0)(k + \lambda c_0 - 1/\beta) + \lambda c_0/\beta$$

where c_j and r_j are the coefficients of the Taylor expansions for the functions $c(\zeta_1)$ and $R(\zeta_1)$. It can be shown that $\Delta_k(\lambda) \neq 0$ for $\sigma > 0$. Therefore the series (5.5) represents a function analytic for $|\zeta_1| < 1$ and $\sigma > 0$. Knowing (5.5), we can obtain the asymptotic expression for the function w_λ with $\lambda \to \infty$. Let us denote

$$A_k = \lim_{\lambda\to\infty} \lambda a_k, \quad B_k = \lim_{\lambda\to\infty} \lambda b_k$$

The recurrent formulas yield

$$A_n = r_n c_0^{-1}; \quad A_k = c_0^{-1}\left(r_k - \sum_{j=1}^{k-n} c_j A_{k-j}\right), \quad k > n$$

$$B_k = 0, \qquad k \geqslant n$$

from which it follows that the series $\sum_{k\geqslant n} A_k \zeta_1^k$ converges and its sum is equal to $R(\zeta_1)\, c^{-1}(\zeta_1)$.

Thus we have the following asymptotic formula for $\lambda \to \infty$:

$$w_\lambda \sim Q(\zeta)\lambda^{-1}, \quad Q(\zeta) = \sum_{k\geqslant n} i^k Q_k \zeta^{3_k}, \quad \beta_k = \beta_k - 1 \tag{5.6}$$

From (5.6) follow (5.2) and (5.3), as well as an expansion in terms of the elementary solutions with the initial values (4.4).

Since the functions w_λ and z_λ are analytic when $\sigma > 0$, the number a in the expression (5.3) can be chosen to be positive and arbitrarily small, i.e. the straight line along which the integration is performed can be brought as near as we like to the vertical axis. The perturbations on the free surface of the stream grow linearly in t, therefore the latter is unstable.

The experimentally arrived at fact of stability can be explained by considering the "drift" character of the perturbations. Let us consider, for simplicity, the case $\alpha = 0$. We note that the growth of the perturbations takes place at the "Lagrangian" point $z_0 = t$. At the same time, the perturbations at the "Eulerian" point $z_0 = \text{const}$ tend to zero: $z_1 \sim te^{-t}$ and $w_2 \sim e^{-t}$. We can therefore assert that the stream is stable at a finite distance from the slit.

BIBLIOGRAPHY

1. Gurevich, M.I., Theory of Jets of Ideal Fluid. M., Fizmatgiz, 1961.
2. Fox, J.L. and Morgan, G.W., On the stability of some flows of ideal fluid with free surfaces. Quart. Appl. Math., Vol.11, №4, 1954.
3. Keyes, K.M., Hydrodynamic stability as a problem with given initial data. Coll. Hydrodynamic Instability, "Mir", 1964.
4. Curle, N., Unsteady two-dimensional flows with free boundaries. Proc. Roy. Soc., Ser. A., Vol.235, №1202, 1956.
5. Wasov, W., Asymptotic Expansions for Ordinary Differential Equations. N.Y. Interscience, 1965.

Translated by L.K.

UDC 532.526

CERTAIN SOLUTIONS OF THE EQUATIONS OF LAMINAR BOUNDARY LAYER WITH LARGE BLOWING

PMM Vol.36, №4, 1972, pp.647-658

V.N. FILIMONOV
(Sevastopol')

(Received October 4, 1971)

The laminar boundary layer is studied for a binary mixture in the case when large blowing takes place from the streamlined surface. Velocity, concentration and temperature distributions within the boundary layer are obtained, formulas for computing the distance to the "line of spreading" are given and expressions for the velocity, concentration and temperature gradients at the surface of the body related to the magnitude of the blowing, are derived.

It was shown earlier [1] that the concentration and temperature gradients at the separation point on the body decrease exponentially with increasing blowing; the author of [2] obtained the power dependence on the blowing everywhere, except at the separation point. The present paper gives expressions containing both these results and an estimate of the region of validity for each of them.

1. The laminar boundary layer equations for a binary mixture have the form [3]

$$(lf''_{\eta\eta})_\eta' + ff''_{\eta\eta} + \Lambda\left[\frac{\rho_e}{\rho} - (f_\eta')^2\right] = 2\xi\,(f_\eta' f''_{\xi\eta} - f_\xi' f''_{\eta\eta})$$

$$\left(\frac{l}{S}\,c_\eta'\right)_\eta' + fc_\eta' = 2\xi\,(f_\eta' c_\xi' - f_\xi' c_\eta') \qquad (1.1)$$

$$\left(\frac{lc_p}{\sigma}\,\theta_\eta'\right)_\eta' + c_p f\theta_\eta' + \frac{l}{S}(c_{p1} - c_{p2})\,c_\eta'\theta_\eta' + l\,\frac{u_e^2}{T_e}\,(f''_{\eta\eta})^2 =$$

$$= f_\eta'\left(\beta c_p\theta + \Lambda\,\frac{\rho_e}{\rho}\,\frac{u_e^2}{T_e}\right) + 2\xi c_p\,(f_\eta'\theta_\xi' - f_\xi'\theta_\eta')$$

$$\eta = \frac{r^k u_e}{\sqrt{2\xi}}\int_0^y \rho\,dy, \qquad \xi = \int_0^x \rho_e\mu_e u_e r^{2k}\,dx$$